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ON UNIQUENESS OF SOLUTIONS FOR NONLINEAR CREEP PROBLEMS IN SYMMETRIC PRESSURE VESSELS†

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Abstract—Uniqueness is established for positive solutions of a nonlinear integral equation which governs the effective stress in internally loaded spherical and incompressible cylindrical pressure vessels subject to primary or secondary transient creep.

INTRODUCTION

It was shown in [1, 2] that the boundary value problems for transient quasistatic creep in hollow spherical and infinite incompressible cylindrical pressure vessels subject to a radial nondecreasing internal pressure can be reduced to an integral equation of the form

$$\sigma(r,t)=\frac{\beta P(t)}{r^j}+\frac{\mu\beta}{r^j}\int_a^b\left[\int_0^t\sigma^n(\xi,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\frac{\mathrm{d}\xi}{\xi}-\mu\left[\int_0^t\sigma^n(r,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}. \tag{1}$$

Here, the unknown function σ stands for the effective stress, and $P(t)$ is proportional to the prescribed internal pressure. The independent variables r and t denote, respectively, radial distance from the center of the vessel and time, a and b are the internal and external radii of the vessel, and

$$\beta^{-1}\equiv\int_a^b\frac{\mathrm{d}\xi}{\xi^{j+1}} \tag{2}$$

where $j=2$ for cylinders, $j=3$ for spheres.

The quantities n and m are creep constants arising from the well-known [3] primary creep law

$$\epsilon_{ij}^{(c)}=\frac{3K}{2}\int_0^t\frac{\sigma_e^{n-1}}{[\epsilon_e^{(c)}]^m}s_{ij}\,\mathrm{d}\tau \tag{3}$$

relating the creep strains $\epsilon_{ij}^{(c)}$ to the effective creep strain‡ $\epsilon_e^{(c)}$, the effective stress σ_e, and the deviatoric stress components s_{ij}. Notice that, for $m=0$, (3) reduces to the law for secondary creep, and the structure of (1) is greatly simplified. The constant μ depends on both elastic and creep constants and is proportional to $K^{1/(m+1)}$. Thus, in the absence of creep ($K=0$), $\mu=0$ and eqn (1) collapses to the solution of the elastic problem.

For the present analysis and in order to be able to use the results of [1] and [2] on boundedness of solutions of (1), we must assume that

$$0\le m\le n-1,\quad \mu>0 \tag{4}$$

$$0<P(0),\ 0\le\dot{P}(t)\quad(t>0) \tag{5}$$

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‡See [1] for detailed definitions of $\epsilon_e^{(c)}$, etc.

and that $\dot{P}$ is integrable on $[0, \infty]$. Also, for $m > 0$, it is assumed that

$$\sigma(r, t) > 0 \quad (a \le r \le b,\, t > 0). \tag{6}$$

In the case $m = 0$, (6) can be shown to be a consequence of (4) and (5).

With the above assumptions, it was shown in [2] for secondary creep and in [1] for primary creep that *any* positive solution σ of (1) has the following bounds:

$$\beta \frac{P(t)}{b^j} \le \sigma(r, t) \le \beta \frac{P(t)}{a^j} \quad (a \le r \le b,\, t \ge 0). \tag{7}$$

It is the purpose of this note to show that there is only one such solution. The main idea of the proof is a uniqueness argument based on the Gronwall-type inequality of Lemma 2 below. This line of attack was suggested by the work of Lewis Wheeler in [4] on the uniqueness of solutions to the displacement problem of nonlinear dynamic elasticity.

UNIQUENESS THEOREM

We shall need the following two elementary lemmas.

LEMMA 1. *For x_1, x_2, and m positive,*

$$|x_1^{1/(m+1)} - x_2^{1/(m+1)}| \le \frac{1}{m+1} (\min\{x_1, x_2\})^{-m/(m+1)} |x_1 - x_2|. \tag{8}$$

A simple proof of this inequality can be obtained by using a finite Taylor expansion of the function $f(x) \equiv x^{1/(m+1)}$ about the smaller of x_1 and x_2 and the fact that $f'' < 0$ on $(0, \infty)$.

Lemma 2. (Gronwall). *Let $u(t)$ be continuous on $[0, T]$ and suppose that, for some finite constants $C > 0$ and $0 \le \alpha < 1$,*

$$0 \le u(t) \le C t^{-\alpha} \int_0^t u(\tau)\, \mathrm{d}\tau \quad (0 < t \le T). \tag{9}$$

Then

$$u(t) = 0 \quad (0 \le t \le T). \tag{10}$$

This lemma is proved by using an integrating factor to put (9) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \left(\int_0^t u(\tau)\, \mathrm{d}\tau \right) \exp\left[\frac{C t^{1-\alpha}}{\alpha - 1} \right] \right\} \le 0.$$

Theorem. The integral eqn (1) *has at most one continuous, positive solution σ on* $[a, b] \times [0, \infty)$.

Proof. Suppose that σ_1 and σ_2 are both positive solutions of (1) and define

$$\bar{\sigma}(r, t) \equiv \sigma_1(r, t) - \sigma_2(r, t). \tag{11}$$

Then, it suffices to show that

$$u^2(t) \equiv \int_a^b r^{j-1} \bar{\sigma}^2(r, t)\, \mathrm{d}r = 0 \tag{12}$$

on any interval $[0, T]$, $T > 0$.

It follows from (1) and (11) that

$$\begin{aligned} \bar{\sigma}(r, t) = \frac{\mu\beta}{r^j} \int_a^b \left(\left[\int_0^t \sigma_1^n(\xi, \tau)\, \mathrm{d}\tau \right]^{1/(m+1)} - \left[\int_0^t \sigma_2^n(\xi, \tau)\, \mathrm{d}\tau \right]^{1/(m+1)} \right) \frac{\mathrm{d}\xi}{\xi} \\ - \mu \left(\left[\int_0^t \sigma_1^n(r, \tau)\, \mathrm{d}\tau \right]^{1/(m+1)} - \left[\int_0^t \sigma_2^n(r, \tau)\, \mathrm{d}\tau \right]^{1/(m+1)} \right). \end{aligned} \tag{13}$$

If this equation is now multiplied by r^{-1} and integrated from a to b, it will follow, using (2), that

$$\int_a^b \bar{\sigma}(r,t)\frac{\mathrm{d}r}{r}=0 \quad (0\le t<\infty). \tag{14}$$

Because of (14), if we multiply both sides of (13) by $r^{j-1}\bar{\sigma}$ and integrate from a to b, we shall obtain

$$u^2(t)=-\mu\int_a^b r^{j-1}\bar{\sigma}(r,t)\left(\left[\int_0^t \sigma_1{}^n(r,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}-\left[\int_0^t \sigma_2{}^n(r,\tau)\,\mathrm{d}\tau\right]^{1/(m+1)}\right)\mathrm{d}r. \tag{15}$$

Notice that, by virtue of (5) and (7),

$$\min\left\{\int_0^t \sigma_1{}^n(r,\tau)\,\mathrm{d}\tau,\int_0^t \sigma_2{}^n(r,\tau)\,\mathrm{d}\tau\right\}\ge\left[\frac{\beta P(0)}{b^j}\right]^n t. \tag{16}$$

Therefore, we can use (16) and Lemma 1, taking

$$x_i=\int_0^t \sigma_i{}^n(r,\tau)\,\mathrm{d}\tau \quad (i=1,2),$$

to derive from (15) the inequality

$$u^2(t)\le\frac{\mu}{m+1}\left(\frac{\beta P(0)}{b^j}\right)^{-mn/(m+1)} t^{-m/(m+1)}\int_a^b r^{j-1}|\bar{\sigma}(r,t)|\int_0^t|\sigma_1{}^n(r,\tau)-\sigma_2{}^n(r,\tau)|\,\mathrm{d}\tau\,\mathrm{d}r. \tag{17}$$

Due to (7) and the continuity of P, it is clear that there exists a finite constant $M=M(T)$ such that

$$|\sigma_1{}^n(r,\tau)-\sigma_2{}^n(r,\tau)|\le M|\bar{\sigma}(r,\tau)| \quad (0\le\tau\le T). \tag{18}$$

In fact, M can even be chosen independent of T if P is bounded. Therefore, if we set

$$C=\frac{\mu M}{m+1}\left(\frac{\beta P(0)}{b^j}\right)^{-mn/(m+1)}, \quad \alpha=\frac{m}{m+1}, \tag{19}$$

it will follow from (17) and (18) that

$$u^2(t)\le Ct^{-\alpha}\int_0^t\int_a^b r^{j-1}|\bar{\sigma}(r,t)||\bar{\sigma}(r,\tau)|\,\mathrm{d}r\,\mathrm{d}\tau. \tag{20}$$

Therefore, by Schwarz's inequality,

$$\begin{aligned} u^2(t)&\le Ct^{-\alpha}\left[\int_a^b r^{j-1}\bar{\sigma}^2(r,t)\,\mathrm{d}r\right]^{1/2}\int_0^t\left[\int_a^b r^{j-1}\bar{\sigma}^2(r,\tau)\,\mathrm{d}r\right]^{1/2}\mathrm{d}\tau\\ &\le Ct^{-\alpha}u(t)\int_0^t u(\tau)\,\mathrm{d}\tau. \end{aligned}$$

This result, together with Lemma 2, completes the proof.

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